PHYSICS 231
2013/14 Term 1

Final Examination

Name: Jake Bobowski
Student Number: Sol'ns

Free Response: Write out complete answers to the following questions. Show your work.

(10pts) **1.** You have three resistors with specified resistances and uncertainties: $R_1 \pm \delta R_1$, $R_2 \pm \delta R_2$, and $R_3 \pm \delta R_3$.

10:10

(a) If the three resistors are connected in series, what is the equivalent resistance $R_s \pm \delta R_s$? Find expressions for R_s and δR_s in terms of R_1, R_2, R_3 and their uncertainties.

1

$$\boxed{R_s = R_1 + R_2 + R_3} \qquad \frac{\partial R_s}{\partial R_1} = 1 \quad \text{likewise for 2 \& 3}$$

2

$$\boxed{\delta R_s = \sqrt{\delta R_1^2 + \delta R_2^2 + \delta R_3^2}}$$

(b) If the three resistors are connected in parallel, what is the equivalent resistance $R_p \pm \delta R_p$? Find expressions for R_p and δR_p in terms of R_1, R_2, R_3 and their uncertainties.

2

$$\boxed{R_p = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}} \qquad \frac{\partial R_p}{\partial R_1} = -\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-2}\left(-\frac{1}{R_1^2}\right)$$

2

$$\boxed{\delta R_p^2 = \frac{\dfrac{\delta R_1^2}{R_1^4} + \dfrac{\delta R_2^2}{R_2^4} + \dfrac{\delta R_3^2}{R_3^4}}{\left(\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}\right)^4}}$$

10 pts

(c) Suppose you want to make a 300 Ω resistor. Given the limited equipment that you have in the lab, your options are to combine three 100 $\Omega \pm 5\%$ resistors in series or to combine three 900 $\Omega \pm 5\%$ resistors in parallel. Compare the resulting numerical values of δR_{s} and δR_{p}.

series:
$$R_1 = R_2 = R_3 = 100\,\Omega \equiv R$$
$$\delta R = (0.05)(100\,\Omega) = 5\,\Omega$$

$$\delta R_s = \sqrt{3\,\delta R^2} = \sqrt{3}\,\delta R = 8.66\,\Omega$$ 1

parallel:
$$R_1 = R_2 = R_3 = 900\,\Omega \equiv R$$
$$\delta R = (0.05)(900\,\Omega) = 45\,\Omega$$

$$\delta R_p^2 = \frac{\frac{3\,\delta R^2}{R^4}}{\left(\frac{3}{R}\right)^4} = \frac{\delta R^2}{3^3}$$

$$\therefore \delta R_p = \frac{\delta R}{3^{3/2}} = \frac{45\,\Omega}{3^{3/2}} = 8.66\,\Omega$$

$$\therefore \boxed{\delta R_s = \delta R_p}$$ 2

Both combination result in the same final ~~error~~ uncertainty $\delta R_{eq} = 8.66\,\Omega$

Notice that the relative uncertainty $\frac{\delta R_{eq}}{R_{eq}} = 2.89\% < 5\%$!

0 pts

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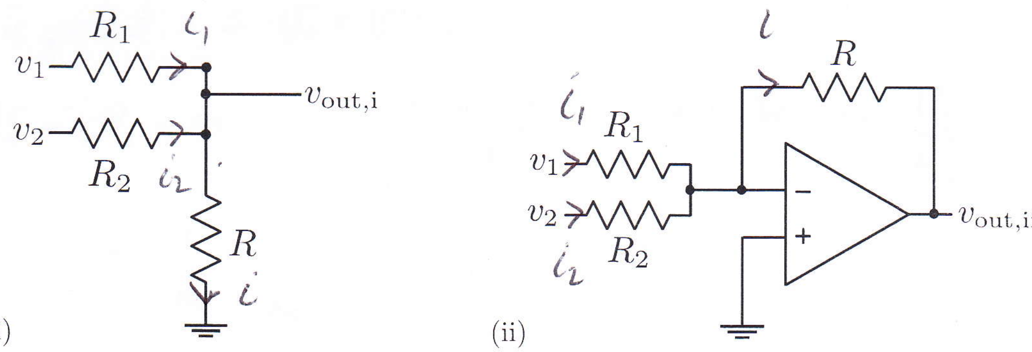
(10pts) **2.** Consider the following two circuits:



(a) For circuit (i), find and expression for $v_{\text{out,i}}$ in terms of v_1, v_2, R_1, R_2, and R.

$$V_1 - i_1 R_1 = V_{out} \quad \therefore \quad i_1 = \frac{V_1 - V_{out}}{R_1}$$

likewise $$i_2 = \frac{V_2 - V_{out}}{R_2}$$

$$i = i_1 + i_2 = \frac{V_1}{R_1} + \frac{V_2}{R_2} - V_{out}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$V_{out} = iR = \left(\frac{V_1}{R_1} + \frac{V_2}{R_2}\right)R - V_{out}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)R$$

$$\cancel{\Rightarrow V_{out}\left[1 + \frac{R}{R_1} + \frac{R}{R_2}\right]} \quad \therefore \frac{V_{out}}{R} = \left(\frac{V_1}{R_1} + \frac{V_2}{R_2}\right) - V_{out}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\therefore V_{out}\left(\frac{1}{R} + \frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2}$$

$$\boxed{\therefore V_{out} = \frac{\frac{V_1}{R_1} + \frac{V_2}{R_2}}{\frac{1}{R} + \frac{1}{R_1} + \frac{1}{R_2}}}$$

4

10 pts

sol'ns

(b) For circuit (ii), find and expression for $v_{\text{out,ii}}$ in terms of v_1, v_2, R_1, R_2, and R.

v_+ is gnded. $\therefore v_+ = v_- = 0$

$\therefore v_1 - i_1 R_1 = 0 \quad \therefore i_1 = \frac{v_1}{R_1}$ likewise $i_2 = \frac{v_2}{R_2}$

$i = i_1 + i_2 = \frac{v_1}{R_1} + \frac{v_2}{R_2}$

$0 - iR = v_{out}$

$\therefore \boxed{-\left(\frac{v_1}{R_1} + \frac{v_2}{R_2}\right)R = v_{out}}$

4

(c) Finally, in your expressions for $v_{\text{out,i}}$ and $v_{\text{out,ii}}$ from parts (a) and (b) set $R_1 = R$ and $R_2 = R$. Simplify your answers as much as possible and write down the resulting $v_{\text{out,i}}$ and $v_{\text{out,ii}}$ expressions.

If $R_1 = R \quad R_2 = R$

$v_{out,i} = \frac{\frac{v_1 + v_2}{R}}{\frac{3}{R}} = \boxed{\frac{v_1 + v_2}{3}}$

If $R_1 = R \quad R_2 = R$

$v_{out,ii} = -\left(\frac{v_1 + v_2}{R}\right)R$

$= \boxed{-(v_1 + v_2)}$

both summing circuits, but (i) has $\frac{1}{3}$ factor & (ii) is inverting.

2

0 pts

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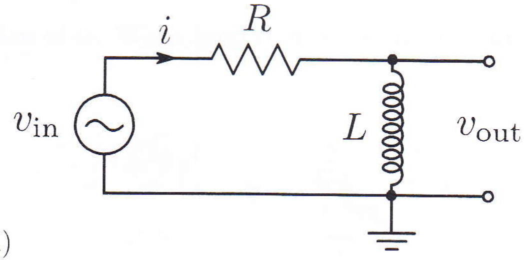
(15pts) **3.** Consider the LR-series circuit shown below:



(i)

(a) If the input voltage is given by $v_{\text{in}} = V_0 \sin \omega t$, what are the amplitude I_0 and phase ϕ of the current i?

1 $I_0 = \frac{V_0}{|Z|}$ $\tan\phi = -\frac{Im[Z]}{Re[Z]}$ 1

For R-L series combo

$Z = R + j\omega L$ $Im[Z] = \omega L$ $Re[Z] = R$

$|Z|^2 = (R + j\omega L)(R - j\omega L) = R^2 + (\omega L)^2$

$\therefore |Z| = R\left(1 + \left(\omega \frac{L}{R}\right)^2\right)^{1/2} = R\sqrt{1 + \left(\omega \frac{L}{R}\right)^2}$

$$\therefore \boxed{I_0 = \frac{V_0/R}{\sqrt{1 + \left(\omega \frac{L}{R}\right)^2}}}$$ 2

$$\boxed{\tan\phi = -\omega \frac{L}{R}}$$ 1

15 pts

(b) For circuit (i) on the previous page, find an expression for $\left|\frac{v_{\text{out}}}{v_{\text{in}}}\right|$ in terms of ω, R, and L. Sketch $\left|\frac{v_{\text{out}}}{v_{\text{in}}}\right|$ as a function of ω. What kind of filter is this circuit?

$$V_{in} = iZ \qquad \therefore \frac{V_{out}}{V_{in}} = \frac{Z_L}{Z} \quad 1$$

$$V_{out} = iZ_L$$

$$\therefore \left|\frac{V_{out}}{V_{in}}\right| = \frac{|Z_L|}{|Z|} = \frac{\omega L}{R\sqrt{1+(\omega L/R)^2}} = \boxed{\frac{\omega L/R}{\sqrt{1+(\omega L/R)^2}}} \quad 2$$

$$\omega \to 0 \qquad \left|\frac{V_{out}}{V_{in}}\right| \to \frac{0}{1} = 0$$

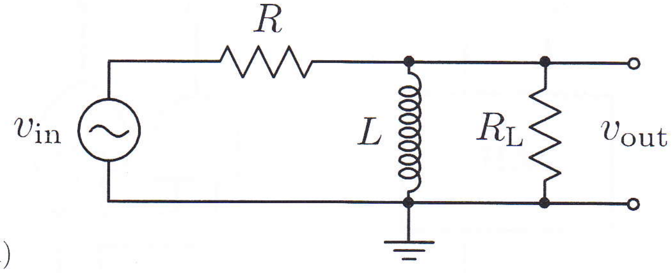
$$\omega \to \infty \qquad \left|\frac{V_{out}}{V_{in}}\right| \to \frac{\omega L/R}{\omega L/R} = 1$$

2

0 pts

Sol'ns

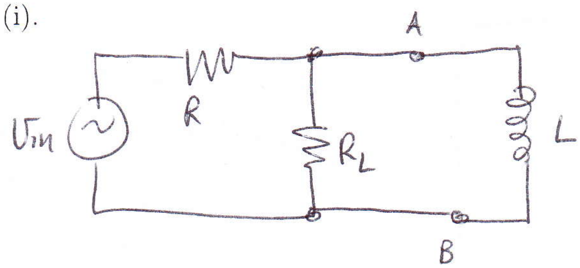
(c) If an oscilloscope is connected across the inductor, the input resistance R_L of the oscilloscope is placed in parallel with the inductor as shown below in circuit (ii):



(ii)

For this modified circuit, what is the new expression for $\left|\frac{v_{out}}{v_{in}}\right|$? It is not necessary to do complicated calculations to find the appropriate expression. Instead, try coming up with an equivalent replacement that turns this back into a series circuit similar to the one shown in figure (i).

Redraw as

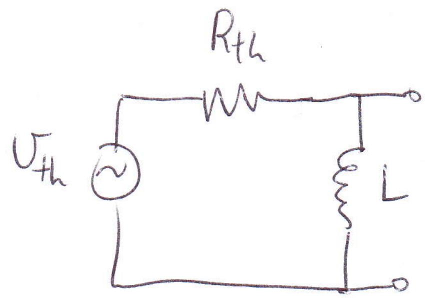


Replace everything to left of AB w/ Thevenin equiv.

$$V_{th} = V_{in}\frac{R_L}{R+R_L} \qquad R_{th} = \frac{RR_L}{R+R_L}$$

→ voltage divider

or $\frac{1}{R_{th}} = \frac{1}{R} + \frac{1}{R_L}$



just like circuit (i)

$$\left|\frac{V_{out}}{V_{th}}\right| = \frac{\omega L/R_{th}}{\sqrt{1+(\omega L/R_{th})^2}}$$

2

$$\therefore \left|\frac{V_{out}}{V_{in}}\right| \cdot \frac{1}{\left(\frac{R_L}{R+R_L}\right)} = \frac{\omega L\left(\frac{1}{R}+\frac{1}{R_L}\right)}{\sqrt{1+\left[\omega L\left(\frac{1}{R}+\frac{1}{R_L}\right)\right]^2}}$$

$$\boxed{\therefore \left|\frac{V_{out}}{V_{in}}\right| = \frac{R_L}{R+R_L}\,\frac{\omega L\left(\frac{1}{R}+\frac{1}{R_L}\right)}{\sqrt{1+\left[\omega L\left(\frac{1}{R}+\frac{1}{R_L}\right)\right]^2}}}$$

3

0 pts

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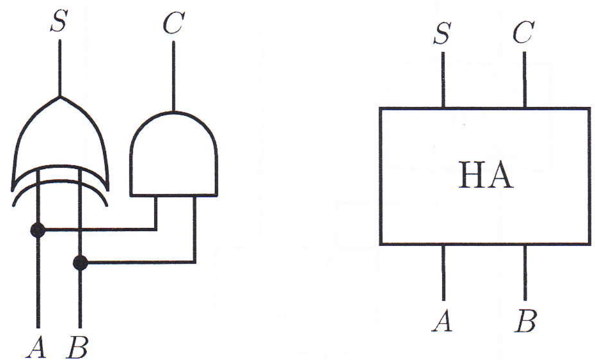
Sol'ns

(10[pts]) **4.** Recall the half-adder circuit used to add to single-bit binary numbers which has two inputs A and B and two outputs S and C:



(a) Write down the truth table for the half-adder circuit.

AND

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

XOR

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

2.5

Half Adder

A	B	S	C
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

(b) Fill in the missing data on the truth table given on the next page. What kind of operation is this circuit performing on the pair of 2-bit binary inputs (A_1, A_0) and (B_1, B_0)?

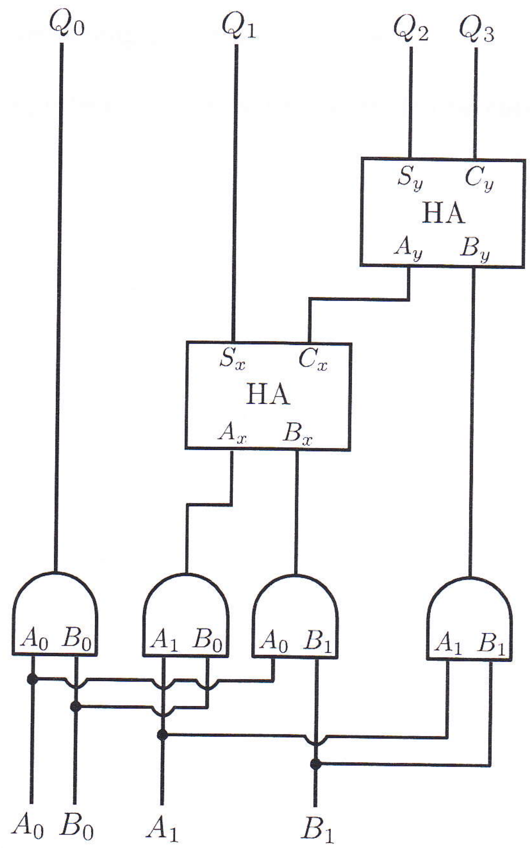
This circuit multiplies two 2-bit binary inputs $(A_1 A_0) \times (B_1 B_0)$.

2.5

10 pts

sol'ns

0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
10	1010



	A_1	A_0	B_1	B_0	A_x	B_x	S_x	C_x	A_y	B_y	S_y	C_y	Q_3	Q_2	Q_1	Q_0	
(0)(0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0)(1)	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(0)(3)	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(2)(0)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1)(1)	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1
(2)(2)	1	0	1	0	0	0	0	0	0	1	1	0	0	1	0	0	4
(3)(2)	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	6
(3)(3)	1	1	1	1	1	1	0	1	1	1	0	1	1	0	0	1	9

5

0 pts

sol'ns

(10pts) **5.** Use Euler's equation $\left(e^{\pm j\phi} = \cos\phi \pm j\sin\phi\right)$ to derive the following two trigonometric identities:

$$\cos^2\phi = \frac{1+\cos 2\phi}{2}$$

$$\sin^2\phi = \frac{1-\cos 2\phi}{2}$$

Note $e^{j\phi} + e^{-j\phi} = 2\cos\phi$

$\therefore \cos\phi = \dfrac{e^{j\phi}+e^{-j\phi}}{2}$

$\therefore \cos^2\phi = \frac{1}{4}\left(e^{j2\phi} + e^{-j2\phi} + 2\right)$

$= \frac{1}{2}\left(\dfrac{e^{j2\phi}+e^{-j2\phi}}{2} + 1\right)$

$$\boxed{\therefore \cos^2\phi = \frac{1}{2}\left(1+\cos 2\phi\right)}$$

Note $e^{j\phi} - e^{-j\phi} = 2j\sin\phi$

$\therefore \sin\phi = \dfrac{e^{j\phi}-e^{-j\phi}}{2j}$

$\sin^2\phi = \dfrac{1}{4j^2}\left(e^{j2\phi} + e^{-j2\phi} - 2\right)$

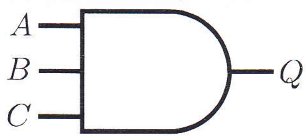
$= \frac{1}{2}\left(1 - \left(\dfrac{e^{j2\phi}+e^{-j2\phi}}{2}\right)\right)$

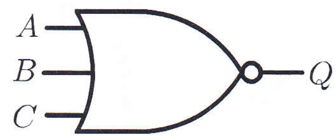
$$\boxed{\therefore \sin^2\phi = \frac{1}{2}\left(1-\cos 2\phi\right)}$$

10 pts

Solns

(10pts) **6.** In class we talked only about logic gates with only two inputs, however, it is possible to make some logic gates with any number of inputs. A 3-input gate AND and a 3-input NOR gate are shown below:





The truth for the 3-input AND gate is:

A	B	C	Q
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

(a) Write down the truth table for the 3-input NOR gate.

For a 3-input OR gate ~~the~~ truth table would be

A	B	C	Q
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

3

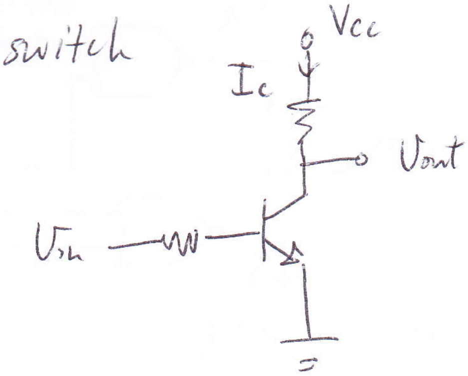
The 3-input NOR gate is just the inverse output

A	B	C	Q
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

10 pts

(b) Design a 3-input NOR gate using only transistors and resistors. *Hint*: One common design uses three transistors.

Recall ~~tr~~ transistor switch



V_{in} LO:

$I_c \approx 0$

$V_{out} = V_{cc} - \cancel{I_c} R$

$= V_{cc}$

$\therefore V_{out}$ is HI.

V_{in} HI:

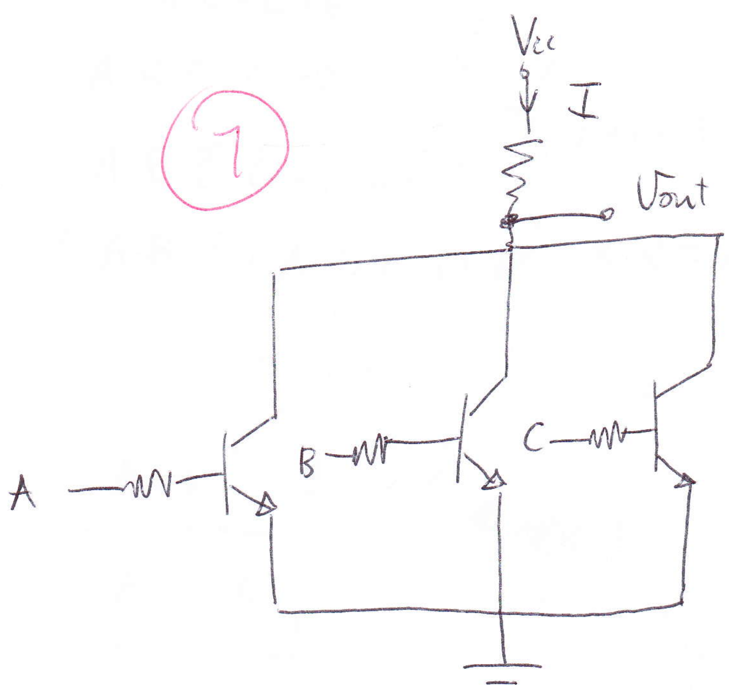
I_c large

transistor forward biased

$V_{out} \approx 0.2\,V \Rightarrow$ LO.

Put transistor switches in parallel

7



I is large anytime one or more transistors conducting $\therefore V_{out} \to$ LO

$V_{out} \to$ HI only if all transistors in off state.

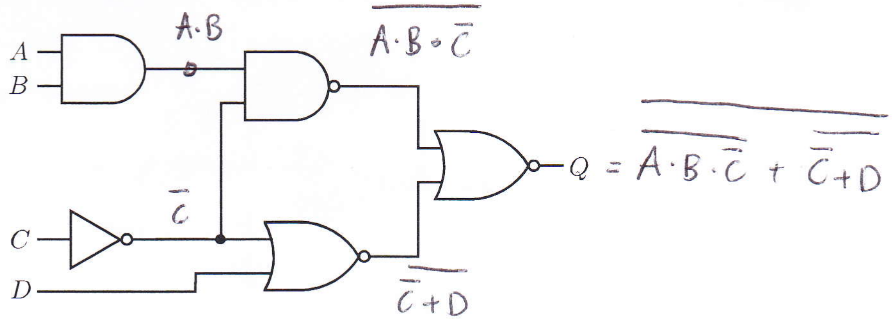
A	B	C	V_{out}
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

✓

0 pts

soln's

(10pts) **7.** Consider the digital circuit shown below:



(a) Write down the logic expression for the output Q. (For example, recall that the logic expression for the X AND Y operation is $X \cdot Y$.)

$$Q = \overline{\overline{A\cdot B\cdot \overline{C}} + \overline{\overline{C}+D}}$$

3

(b) Simplify the expression for Q obtained in part (a) as much as possible. For part of your solution, you may find De Morgan's theorems helpful: $\overline{A\cdot B} = \overline{A}+\overline{B}$ and $\overline{A+B} = \overline{A}\cdot\overline{B}$. Draw the simplified digital circuit. Are any of the inputs irrelevant to the state of the output Q?

$$Q = \overline{\overline{A\cdot B\cdot \overline{C}} + \overline{\overline{C}+D}}$$

$$= \overline{\overline{A\cdot B\cdot \overline{C}}}\cdot\overline{\overline{\overline{C}+D}} \quad \text{De Morgan}$$

$$= A\cdot B\cdot \overline{C}\cdot(\overline{C}+D) \quad \overline{\overline{X}} = X$$

$$= A\cdot B\cdot \overline{C}\cdot \overline{C} + A\cdot B\cdot \overline{C}\cdot D \quad \text{distributive}$$

$$= A\cdot B\cdot \overline{C} + A\cdot B\cdot \overline{C}\cdot D \quad X\cdot X = X$$

X	X·X
0	0
1	1

5

$$= A\cdot B\cdot \overline{C}\cdot(1+D) \quad X\cdot 1 = X$$

X	1	X·1
0	1	0
1	1	1

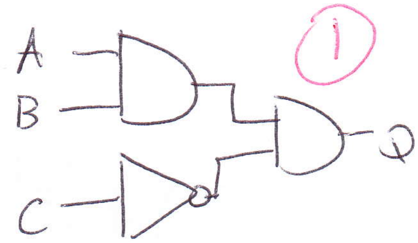
$$= A\cdot B\cdot \overline{C}\cdot 1$$

$$\boxed{\therefore Q = A\cdot B\cdot \overline{C}}$$

$1 + X = 1$

X	1	X+1
0	1	1
1	1	1

1



equivalent circuit.

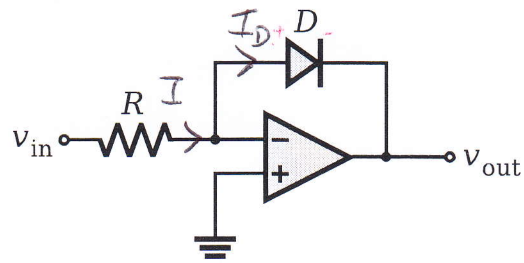
1

Input D irrelevant to state of output Q.

10 pts

Sol'ns

(10pts) **8.** Consider the so-called "log amplifier" shown below:



Recall that the current in a diode is given by $I_D = I_0\left(e^{eV_D/k_BT} - 1\right)$ where I_0 is a constant and V_D is the voltage across the diode. Assume that v_{in} and R are chosen such that $I_D \gg I_0$, show that:

$$v_{out} = G\ln\left(\frac{v_{in}}{RI_0}\right)$$

Find an expression for the proportionality constant G.

$$I_D = I_0\left(e^{eV_D/k_BT} - 1\right) = I_0 e^{eV_D/k_BT} - I_0$$

if $I_D \gg I_0$, then $I_D \approx I_0 e^{eV_D/k_BT}$

v_+ gnded $\therefore v_+ = v_- = 0$.

$\therefore v_{in} - IR = 0 \quad \therefore I = \frac{v_{in}}{R}$

current into op amp is zero. $\therefore I = I_D$

$\therefore \frac{v_{in}}{R} \approx I_0 e^{eV_D/k_BT}$

Note also that $\cancel{v_-}^{\,0} - V_D = V_{out}$

$\therefore v_{out} = -V_D$

Solve for V_D

$$\ln\left(\frac{v_{in}}{I_0R}\right) = \frac{eV_D}{k_BT}$$

$\therefore V_D = \frac{k_BT}{e}\ln\left(\frac{v_{in}}{I_0R}\right)$

$\therefore \boxed{v_{out} = -\frac{k_BT}{e}\ln\left(\frac{v_{in}}{I_0R}\right) \qquad G = -\frac{k_BT}{e}}$

$v_{out} \propto \ln v_{in}$

$\rightarrow$ Log Amp.

10 pts